Granular Computing:

Methodology and Mathematical Framework

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1. Context of Granular Computing
2. Defining information granules
3. Granular Worlds
4. Sets and intervals
5. Fuzzy sets

Spectrum of Intelligent Systems

speech recognition

intelligent databases

fuzzy controller

autonomous robot

Web data mining

intelligent highway

Web agent

intelligent traffic controller

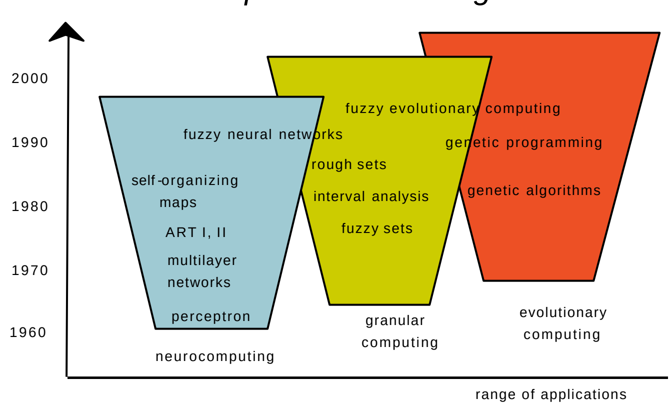
user-friendly appliances

evolvable hardware

character recognition

Information processing technologies

Computational Intelligence



Defining Computational Intelligence

... a system is *computationally intelligent* when it: deals only with numerical (low-level) data, has

a pattern recognition component, does not use knowledge in the AI sense...

J.C. Bezdek, On the relationship between neural networks, pattern recognition and intelligence, J. Approximate Reasoning, 6, 1992, 85-107

Defining Computational Intelligence

... CI substitutes intensive computation for insight into how the system works. NNs, FSs and EC were all shunned by classical system and control theorists. CI umbrellas and unifies these and other revolutionary methods

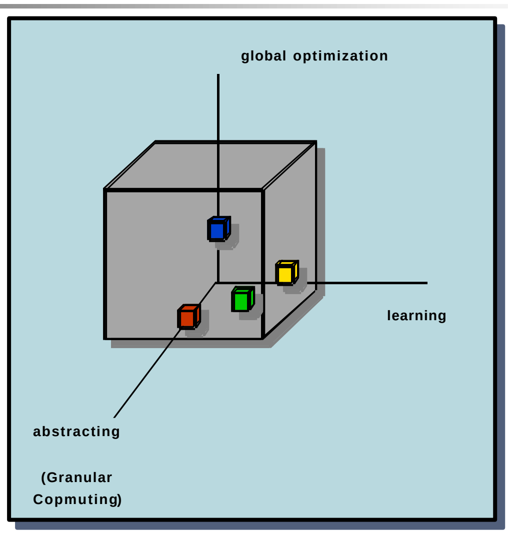
W.J.Karplus, 1996

Defining
Computational Intelligence

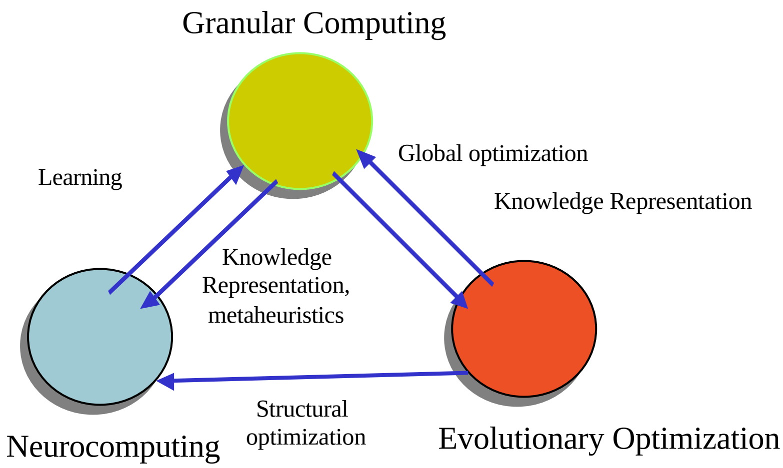
Computational Intelligence is a research endeavor aimed at conceptual and algorithmic integration of technologies of granular computing, neural networks and evolutionary computing

W. Pedrycz, <u>Computational Intelligence: An Introduction</u>,1997

Dimensions of
Computational Intelligence



Synergy of components of Computational Intelligence



Information granules are everywhere…

Spacial granulation: Image processing

identification of segments (regions) and finding relationships between them

Temporal granulation: Signal processing

sampling signals in time and quantization in phase-space (discretization)

System modelling

System partitioning, human-centred interpretation, rule-based models, qualitative models…

Granular Computing

Computing based on information granules: basic entities recognized in problem description

Information granules: composed of elements drawn together owing to their similarity, functional closeness, spatial neighborhood, etc

Information granulation: a process of developing Information granules

Formal frameworks for information granulation

$$\underbrace{A}_{\text{granule}} : \mathbf{X} \rightarrow \underbrace{G}_{\text{framework}}(\mathbf{X})$$

G : Set theory, interval analysis

G : Fuzzy sets

G : Rough sets

G : Probabilistic granules

Granular Computing:
Set Theory and Interval Analysis

• support basic processes of abstraction by employing an idea of dichotomization
• two-valued logic as a formal means of computing
• basic mechanism of abstraction
• information hiding
• level of specificity of information granules captured by set cardinality

Granular Computing:
Fuzzy Sets

• departure from dichotomization (*yes*-*no*)
• refinement of concepts by accepting *continuous* membership grades (distinguishability through membership)
• based on ideas of multivalued (fuzzy) logic
• mechanism of abstraction capturing *qualitative* as well as *quantitative* facet of concepts

Granular Computing: Non-Aristotelian View

..in analyzing the Aristotelian codification, I had to deal with the two-valued, "either-or" type of orientation. In living, many issues are not so sharp, and therefore a system that posits the general sharpness of "either-or" and so objectifies "kind" , is unduly limited; it must be revised and more flexible in terms of "degree"…

A. Korzybski, 1933

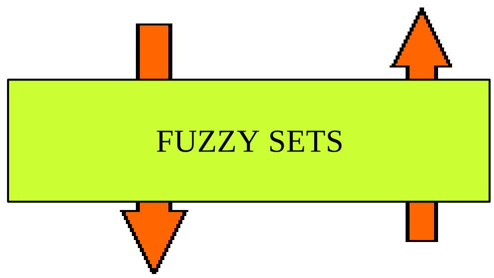
Granular Computing: Non-Aristotelian View

…it is a paradox, whose important familiarity fails to diminish, that the most highly developed and useful scientific theories are ostensibly expressed ion terms of objects never encountered in experience. And the "point-planet" of astronomy, the "perfect gas" or thermodynamics or the "pure species" of genetics are equally remote from exact realization…

M. Black, 1938

“Impedence” Mismatch

Designer/User: linguistic terms, design objectives, conflicting requirements

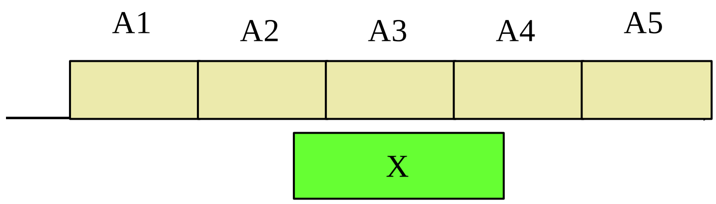


Computer Systems: two-valued logic

Granular Computing: Rough Sets

- defining information granules through their lower and upper bounds
- identifying regions with a lack of knowledge about concept
- expressing aspects of uncertainty through “rough” boundaries

Granular Computing: Rough Sets



RoughSet(X, **A**)=(lower approximation, upper approximation)

Lower approximation ={A3}

Upper approximation ={A2, A3, A4}

Granular Computing: Probabilistic granules

- granules represented by some probability functions (probability density functions)
- probabilistic relationships between elements in information granules
- calculus of probabilities as a processing environment

Characterising granularity

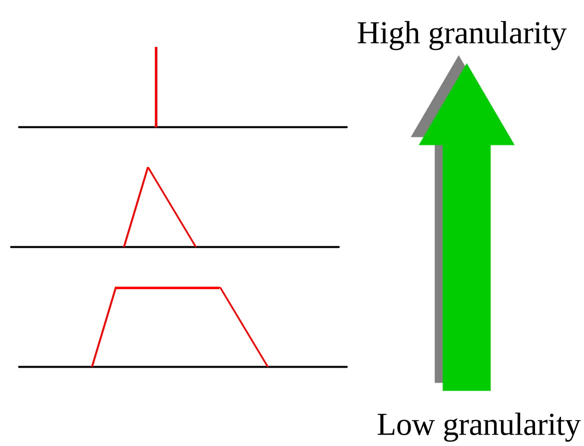
Measuring "size" of information granules

Smaller information granules – *higher* granularity (specificity)

Depends on the formal model of information granulation

<u>Sets</u>: cardinality
<u>Fuzzy sets</u>: σ-count
<u>Probabilistic granules</u>: variance (st. deviation)

Characterising granularity

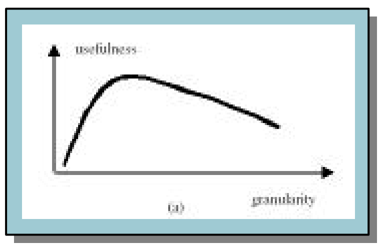


Characterising granularity

Size vs. specificity

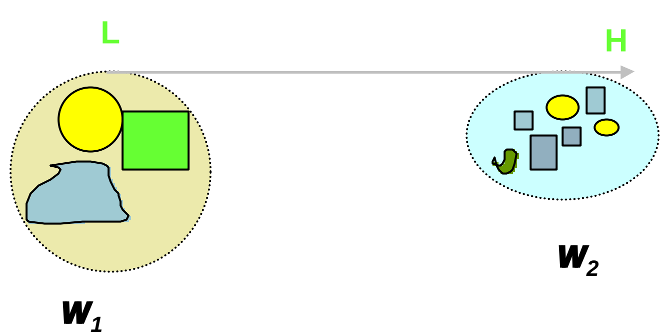
$$\text{Card}\,(A) = \int_X A(\mathbf{x})d\mathbf{x}$$

Usefulness



Granular Worlds

granularity

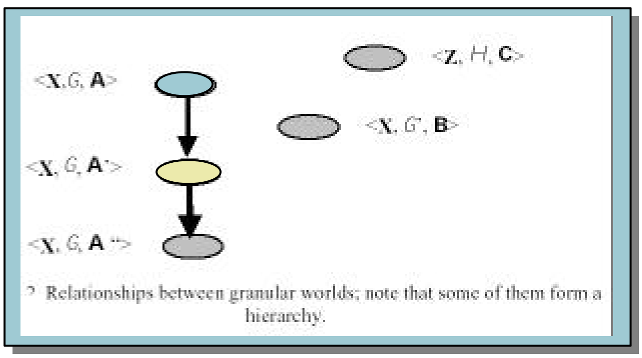


Qualitative Models ⟷ *Quantitative Models*

Defining a Granular World

$$\mathbf{G} = \langle \mathbf{X}, G, \mathbf{A}, \ldots \rangle$$

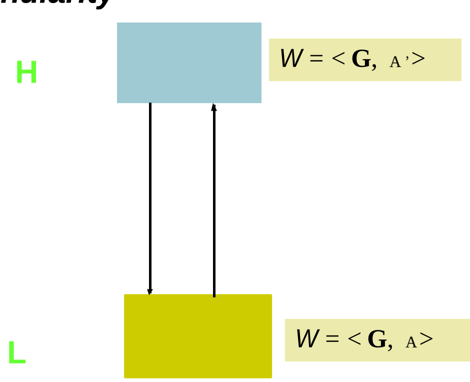
Granular world — data space — granulation framework — family of granules



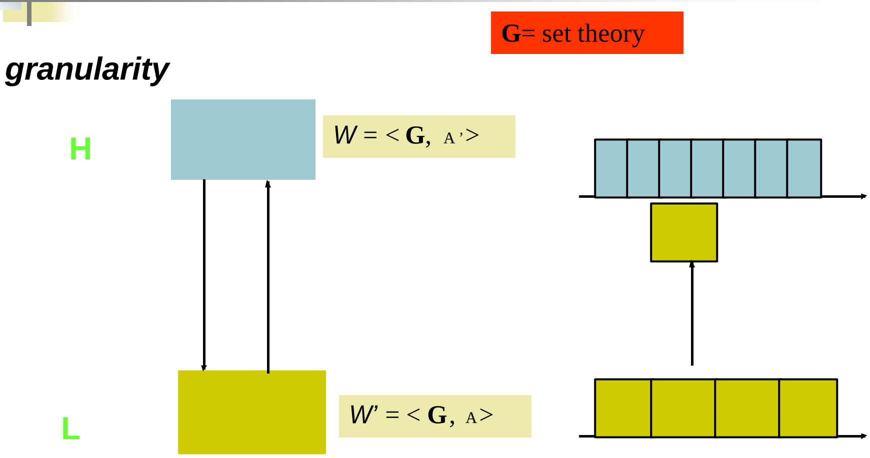
? Relationships between granular worlds; note that some of them form a hierarchy.

Communication between granular worlds

granularity



Communication between granular worlds



Defining a Granular World

$$\mathbf{G} = \langle \mathbf{X}, G, \mathbf{A}, \mathbf{C} \rangle$$

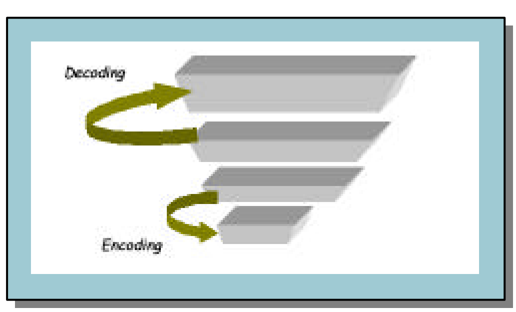
Granular world

data space

granulation framework

family of granules

communication mechanism



Example of an instance of Granular Computing

Sets:

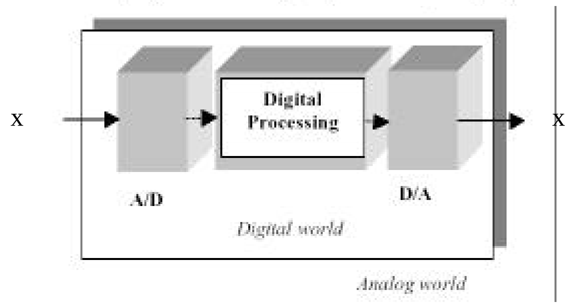
A/D: $Enc(X) : X = \{x\} \in \mathbf{R} \rightarrow X \in \mathcal{P}(\mathbf{R})$

D/A: $Dec(X) : X \in \mathcal{P}(\mathbf{R}) \rightarrow X = \{x'\} \in \mathbf{R}$

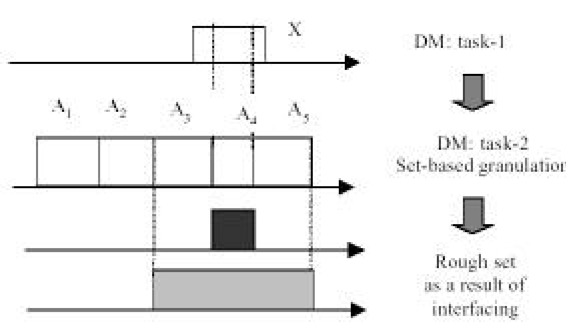
Fuzzy Sets:

A/D: $Enc(X) : X = \{x\} \in \mathbf{R} \rightarrow X \in \mathcal{F}(\mathbf{R})$

D/A: $Dec(X) : X \in \mathcal{P}(\mathbf{R}) \rightarrow X = \{x'\} \in \mathbf{R}$



Example of interoperability of DM tasks realized in two different granular worlds



Rough Set is an outcome of interfacing two Data Mining tasks with different granule sizes.

Main Points

- Granular Computing as an essential component of Computational Intelligence
- Diversity of fundamental technologies underlying GC (sets, fuzzy sets, rough sets…)
- Open fundamental problems of communication between granular worlds

Mathematical Framework: Sets and Intervals

Historical Background

… all mathematical theories can be regarded as extensions of the general theory of sets … on these foundations I can state that I can build up the whole of the mathematics of the present day …

Bourbaki (1948), L'architecture des mathematiques, Les Grands Courants de la Pensee Mathematique, F. LeLionnais (ed.)

Historical Background

… despite the remoteness from sense-experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than sense-perception …

Goedel (1940), The consistency of the axiom of choice and of the generalised continuum hypothesis with the axioms of set theory, Princeron University Press

Historical Background

Georg Cantor

foundations of Set Theory 1874-1884

Paradoxes in initial theory:

- The cardinality of “set of all sets” vs. cardinality of the set of all subsets drawn from this set.

- Construction of sets paradox (Russel)

A={X | X is not a member of X}

Historical Background

Modern Set Theory (ies) - (eg. Goedel, 1940)

Abandons a uniform view of sets and introduces some form of “hierarchy”

Emphasises the semantical transformation (qualitative change) that occurs when grouping individual elements into sets or conversely when identifying sub-sets within a given set

Three primitive notions of: class, set and membership

Historical Background

Modern Set Theory (ies)

Class - an entity corresponding to some but not necessarily all properties

(-> various classes represent conceptually different entities)

A class that is a member of some other class is considered a set, otherwise it is considered a proper class

(-> a “set of all sets” is a proper class)

Historical Background

Arythmetic defined on sets (intervals)

- Warmus, 1959, Sunaga, 1958, More, 1962

Guaranteed results of computation

- Hansen, 1975, Nikel, 1981

Granular Computing

- Zadeh, 1979, 1997

The formalism of sets

Explicit specification
of finite sets

$$A=\{a_1, a_2, a_3, \ldots, a_n\}$$

Implicit specification

$$B=\{b \mid b \text{ has properties } P_1, P_2, \ldots, P_n\}$$

Basic properties of sets

Cardinality $|A|$

Set inclusion $A \subset B$

Belonging
(*characteristic function)*

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Complement
(wro. *universal set* X) $\overline{A}$

Basic set operations

Union $A \cup B = \{x \mid x\in A \text{ or } x\in B\}$

Intersection $A \cap B = \{x \mid x\in A \text{ and } x\in B\}$

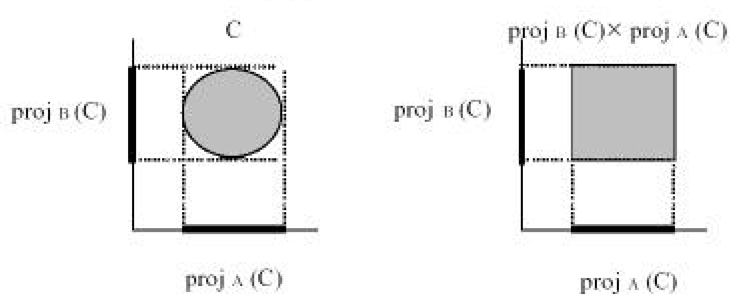
Complementation $A \cup \overline{A} = \mathbf{X}$

Relative complementation: $A \setminus B = \{x \mid x\in A \text{ and } x\notin B\}$
(*subtraction*)

Basic set operations

Cartesian product: $A \times B = \{(x, y) \mid x\in A \text{ and } y\in B\}$

Projection: $\text{proj}_A(C) = \{ x\in A \mid \exists y \text{ for which } (x, y)\in C\}$



Properties of set operations

Involution:	$\overline{\overline{A}} = A$
Distributivity:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Associativity:	$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$
Comutativity:	$A \cup B = B \cup A$ $A \cap B = B \cap A$

Properties of set operations

Absorption:	$(A \cup B) \cap A = A$ $(A \cap B) \cup A = A$
Idempotence:	$A \cup A = A$ $A \cap A = A$
Law o excluded middle:	$A \cup \overline{A} = \mathbf{X}$
Law of contradiction:	$A \cap \overline{A} = \varnothing$
DeMorgan laws:	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Functional mapping of sets

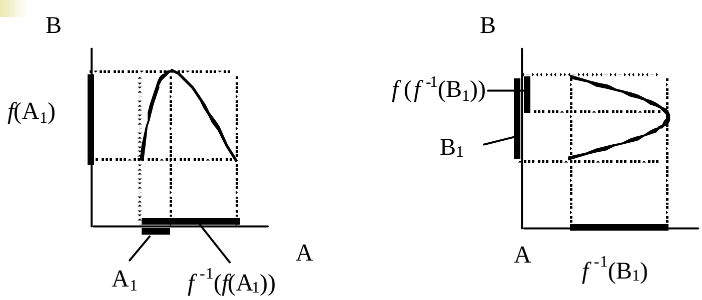
Direct mapping $f{:}A \longrightarrow B$

$$f(A_1)=\{f(x)\,|\,x\in A_1\}=B_1$$

Inverse mapping $f^{-1}: B \longrightarrow A$

$$f^{-1}(B_1)=\{x\in A\,|\,f(x)\in B_1\}$$

Functional mapping of sets



Composition of an image and a reciprocal image of a set

Note that:

$$A_1 \subset f^{-1}(f(A_1))$$

$$f(f^{-1}(B_1)) \subset B_1$$

Functional mapping of sets

Distributivity and monotonicity

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$$
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$
$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$
$$\text{if } A_1 \subset A_2 \text{ then } f(A_1) \subset f(A_2)$$
$$\text{if } B_1 \subset B_2 \text{ then } f^{-1}(B_1) \subset f^{-1}(B_2)$$

Arythmetical operations on sets

General operation *"op "*

$$A_1 - A_1 = \{ x - y \mid x \in A_1, y \in A_1 \} \neq \{0\}$$

e.g. for *"op "* = "-"

$$A_1 - B_1 = \{ x - y \mid x \in A_1, y \in B_1 \}$$

Note that:

$$A_1 - A_1 = \{ x - y \mid x \in A_1, y \in A_1 \} \neq \{0\}$$

Overestimation of results -> *Dependency effect*

Set enclosure

- Need to simplify description of the topologies of sets in order to quantify the overestimate of the image set (i.e. *dependency effect)*

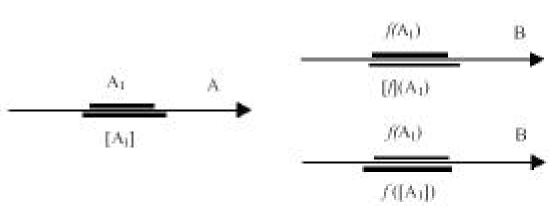
  - Hyperboxes
  - Ellipsoids
  - Polyhedrons

The use of enclosures leads to a secondary overestimation -> *wrapping effect*

Set enclosure

Enclosure of the operator "*op* "

$$A_I\ [op]\ B_I = [\{ x_I\ op\ y_I \mid x_I \in A_I, y_I \in B_I \}]$$



Overestimation of the function image set due to enclosure operation. Note that the function enclosure guarantees that $[f](A_I) \supset f(A_I)$ and the set enclosure guarantees that $f([A_I]) \supset f(A_I)$ but $[f](A_I)$ is in general quite different from $f([A_I])$.

Interval analysis

Definitions

Interval $[x]=\{x \in R \mid x \geq x^- \in R \text{ and } x \leq x^+ \in R\}$

$$x^- = \sup\{a \in \mathbf{R} \mid \forall x \in [x], a \leq x\}$$

$$x^+ = \inf\{a \in \mathbf{R} \mid \forall x \in [x], x \leq a\}$$

$$\text{width}([x]) = x^+ - x^-$$

$$\text{center}([x]) = (x^+ + x^-)/2 = x^- + \text{width}([x])$$

Basic operations on intervals

Union and intersection

$$[x] \cup [y] = \{ a \in \mathbf{R} \mid a \in [x] \text{ or } a \in [y]\}$$

$$[x] \cap [y] = \{ a \in \mathbf{R} \mid a \in [x] \text{ and } a \in [y]\}$$

Interval union (to make intervals closed with respect of union)

$$[x]\,[\cup]\,[y] = [\min(x^-, y^-), \max(x^+, y^+)]$$

Interval subtraction (to make intervals closed with respect of subtraction)

$$[x]\,[\backslash]\,[y] = [[x] \backslash [y]] = [a, b]$$

$$a = \sup\{c \in \mathbf{R} \mid \forall d \in [x]\backslash[y], c \leq d\}$$

$$b = \inf\{c \in \mathbf{R} \mid \forall d \in [x]\backslash[y], d \leq c\}$$

Basic operations on intervals

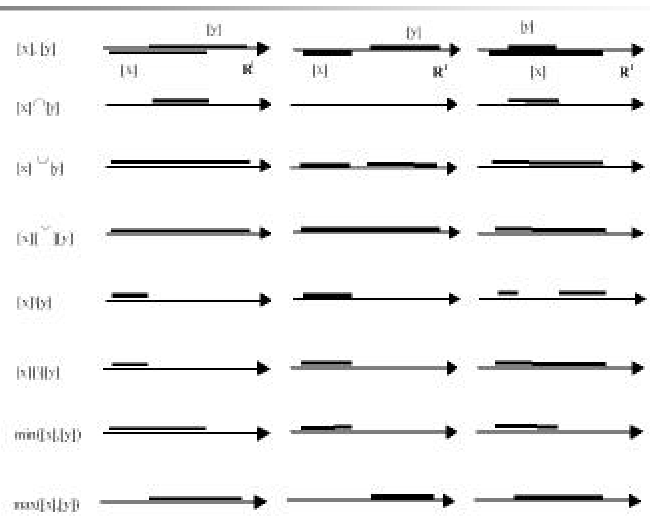
Interval maximum of intervals

$$\max([x],[y])=[\max(x^-,y^-),\ \max(x^+,y^+)]$$

Interval minimum if intervals

$$\min([x],[y])=[\min(x^-,y^-),\ \min(x^+,y^+)]$$

Basic operations on intervals



Arithmetical operations on intervals

Addition of intervals [x] and [y] is
$[x]+[y]=[x^- + y^-, x^+ + y^+]$

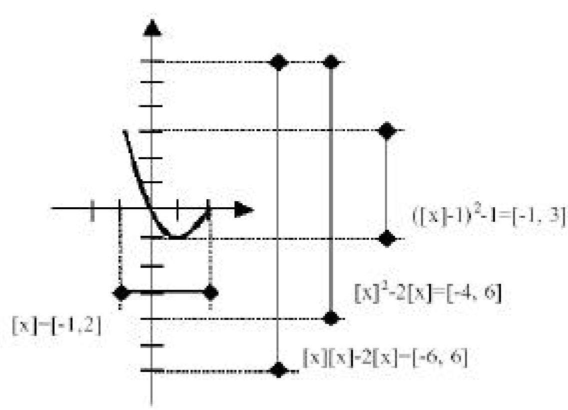
Subtraction of intervals [x] and [y] is
$[x]-[y]=[x^- - y^+, x^+ - y^-]$

Multiplication of intervals [x] and [y] is
$[x][y]=[\min\{x^- y^-, x^- y^+, x^+ y^-, y^+ x^+\}, \max\{x^- y^-, x^- y^+, x^+ y^-, y^+ x^+\}]$

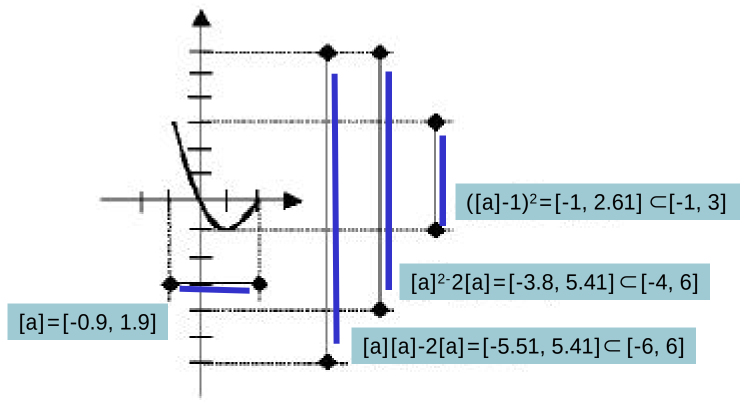
Multiplication of interval [x] by a scalar w
$w[x]=[w x^-, w x^+]$ if $w>0$
$w[x]=[w x^+, w x^-]$ if $w<0$

Division of intervals [x] and [y] is
$[x]/[y]=[x](1/[y])$ where, $1/[y] = [1/y^+, 1/y^-]$ if $y^- > 0$ or $y^+ < 0$

Example of *dependency effect* for operations on intervals



Example of *wrapping effect* for operations on intervals

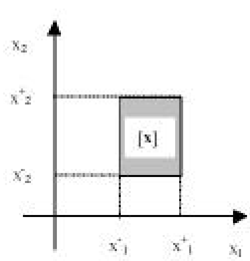


Interval vectors

Definition: $[\mathbf{x}]=[x_1]\times[x_2]\times\ldots\times[x_n]$,

where $[x_i]$ are intervals, $[x_i]=[x_i^-, x_i^+]$, $i=1,\ldots,n$

Example:



Two-dimensional hyperbox, $[\mathbf{x}]\in I(\mathbf{R}^2)$.

Interval vectors

$$[x] \cap [y] = ([x_1] \cap [y_1] \times [x_2] \cup [y_2] \times \ldots \times [x_n] \cap [y_n])^T$$

$$\boxed{[x] \cup [y] = ([x_1] \cup [y_1] \times [x_2] \cup [y_2] \times \ldots \times [x_n] \cup [y_n])^T}$$

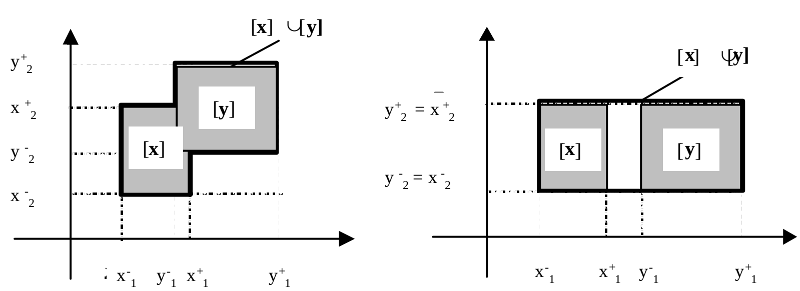
The union and subtraction of hyperboxes produces in general a more complex topology. In order to make hyperboxes closed with respect of union and subtraction operation we define hyperbox union.

$$[x]\,[\cup]\,[y] = [[x] \cup [y]]$$

and hyperbox subtraction

$$[x]\,[\setminus]\,[y] = [[x] \setminus [y]]$$

Interval vectors



A typical and a special case of a union of hyperboxes.

Interval vectors

Arithmetical operations:

$$w[\mathbf{x}]=(w[x_1])\times(w[x_2])\times\ldots\times(w[x_n])$$

$$[\mathbf{x}]^T[\mathbf{y}]=[x_1][y_1]+[x_2][y_2]+\ldots+[x_n][y_n]$$

$$[\mathbf{x}]^T+[\mathbf{y}]=([x_1]+[y_1])\times([x_2]+[y_2])\times\ldots\times([x_n]+[y_n])$$

Operations on the RHS are operations on intervals

Interval matrices

Definition:

$$[\mathbf{A}]=\begin{bmatrix}[a_{11}] & \ldots & [a_{1n}]\\ \ldots & \ldots & \ldots\\ [a_{m1}] & \ldots & [a_{mn}]\end{bmatrix}=[\mathbf{a}_1]\times\ldots\times[\mathbf{a}_n]$$

$[\mathbf{a}_j]$ are interval vectors, $[\mathbf{a}_j]=[a_{1j}]\times\ldots\times[a_{mj}]$, $j=1,\ldots,n$.

Interval matrices

Basic operations:

$$[\mathbf{A}] \cap [\mathbf{B}]=([\mathbf{a}_1] \cap [\mathbf{b}_1] \times [\mathbf{a}_2] \cap [\mathbf{b}_2] \times \ldots \times [\mathbf{a}_n] \cap [\mathbf{b}_n])$$

$$[\mathbf{A}] \cup [\mathbf{B}]=([\mathbf{a}_1] \cup [\mathbf{b}_1] \times [\mathbf{a}_n] \cup [\mathbf{b}_n] \times \ldots \times [\mathbf{a}_n] \cup [\mathbf{b}_n])$$

$$[\mathbf{A}] \setminus [\mathbf{B}]=([\mathbf{a}_1] \setminus [\mathbf{b}_1] \times [\mathbf{a}_n] \setminus [\mathbf{b}_n] \times \ldots \times [\mathbf{a}_n] \setminus [\mathbf{b}_n])$$

To make hyperboxes closed with respect of union and subtraction operation we define hyperbox union and hyperbox subtraction.

$$[\mathbf{A}]\, [\cup]\, [\mathbf{B}]=[[\mathbf{A}] \cup [\mathbf{B}]]$$

$$[\mathbf{A}]\, [\setminus]\, [\mathbf{B}]=[[\mathbf{A}] \setminus [\mathbf{B}]]$$

Interval martices

Arithmetical operations:

$$\mathrm{w}[\mathbf{A}]=(\mathrm{w}[\mathbf{a}_1]) \times (\mathrm{w}[\mathbf{a}_2]) \times \ldots \times (\mathrm{w}[\mathbf{a}_n])$$

$$[\mathbf{A}][\mathbf{B}]=([\mathbf{a}_i]^T[\mathbf{b}_j]),\ 1 \le i \le n,\ 1 \le j \le n$$

$$[\mathbf{A}][\mathbf{x}]=([\mathbf{a}_i]^T[\mathbf{x}]),\ 1 \le i \le n$$

$$[\mathbf{A}]+[\mathbf{B}]=([\mathbf{a}_i]+[\mathbf{b}_i]),\ 1 \le i \le n$$

Operations on the RHS are operations on interval vectors

Interval martices

Note that for interval matrices:

$$([A][B])[C] \neq [A]([B][C])$$

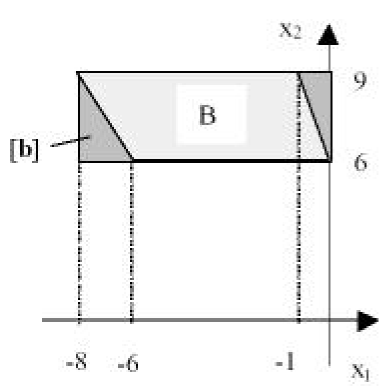
$$[A]([B]+[C]) \neq ([A][B]+[A][C])$$

$$[A]([B]+[C]) \subset ([A][B]+[A][C])$$

Violation of *associativity* is due to *wrapping effect.*
Subdistributivity is due to the *dependency effect* (repetition of [A] leads to an overestimate).

Interval martices

Example of wrapping:



$$[A]=\begin{bmatrix} [-1,1] & [-2,-1] \\ [0,0] & [3,3] \end{bmatrix},\ [x]=\begin{bmatrix} [1,2] \\ [2,3] \end{bmatrix}$$

$$[b]=[A][x]$$

$$B=\{[A]x \mid x\in [x]\}$$

Enclosure of functions

f: $\mathbf{R}^n \rightarrow \mathbf{R}$ differentiable over the interval vector [x]

Natural enclosure of $f([x])$

$$[f]([x])$$

is obtained by substituting arithmetical operations with equivalent interval arythmetic (repetition of [x] leads however to big overestimates of the enclosure due to *dependency effect*)

Enclosure of functions

Example:

$$f(x)=x - x^3/6 + x^5/120, \qquad x[-3, 3]$$

Minimal enclosure $[f]^* = [-1.005, 1.005]$

Natural enclosure

$$[f]([x]) = [x] - [x]^3/6 + [x]^5/120$$

$$= [-3, 3] - [-4.5, 4.5] + [-2.025, 2.025]$$

$$= [9.525, -9.525]$$

Enclosure of functions

f: $\mathbf{R}^n \rightarrow \mathbf{R}$ differentiable over the interval vector $[\mathbf{x}]$

$\mathbf{c}$=center($[\mathbf{x}]$)
$\mathbf{w}$=[width($[x_1]$),…, width($[x_n]$)]

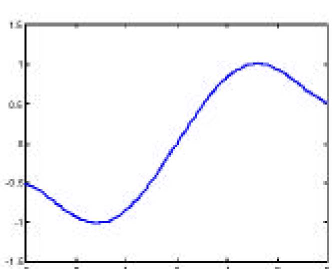
mean value theorem implies $f(\mathbf{x})=f(\mathbf{c})+f'(\mathbf{x})(\mathbf{x}-\mathbf{c})$

Centered enclosure of $f([x])$

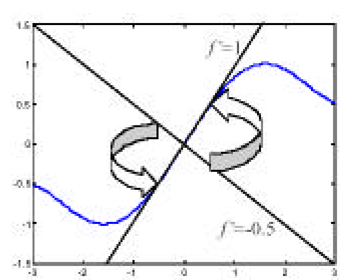
$$[f_c]([\mathbf{x}])=f(\mathbf{c})+[f'] ([\mathbf{x}])[-(\mathbf{w}/2), \mathbf{w}/2]$$

Enclosure of functions

Example:



$f(x)=x - x^3/6 + x^5/120$



$[f_c]([\mathbf{x}])=[-3, 3]$

Enclosure of functions

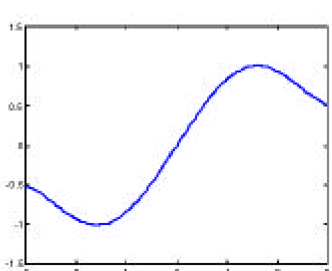
Space subdivision enclosure

$$[\mathbf{x}_1] \cup [\mathbf{x}_2] \cup \ldots \cup [\mathbf{x}_n] = [\mathbf{x}]$$

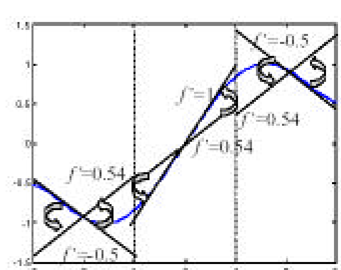
$$[f]([\mathbf{x}]) = [f]([\mathbf{x}_1]) \cup [f]([\mathbf{x}_2]) \cup \ldots \cup [f]([\mathbf{x}_n])$$

Enclosure of functions

Example:



$f(x) = x - x^3/6 + x^5/120$



$[f]([\mathbf{x}]) =$
$= [-1.47, -0.39] \cup [-1, 1] \cup [0.39, 1.47]$
$= [-1.47, 1.47]$

Fuzzy Sets: A Motivation

Formal expression of a concept of continuous boundaries

- Data processing: Making sense of data
- Decision-making and control problems
- Image processing and computer vision

Making sense of data
Data Mining

Databases: information storage and information retrieval using a formalism of query languages

user-friendly interface: allow for linguistic queries and

abstraction/summarization and explanation mechanisms

Control Problems

- Design a controller that maintains a *comfortable* room temperature
- Design a highway traffic control system that assures *safe* driving environment
- Design a system that can *park* a car

Rule-based systems

- Expressing domain knowledge in a form of rules
- -- if condition then conclusion
  - Easy to understand and acquire
  - Modular system
  - Rules are generalizations of existing patterns of decision-making, classification, control,…

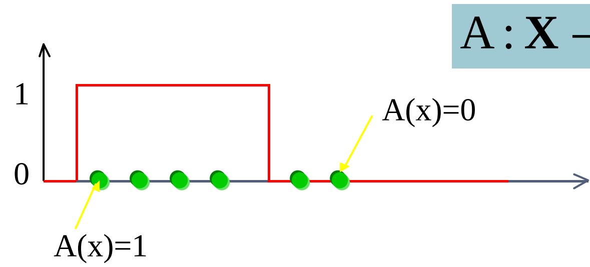
Making sense of images
Granular Art

- Domain-oriented, common sense knowledge (landscape)
- - if a region is *mostly green* and *highly textured* then it's likely to be vegetation
- - if a region is *intensely green* and overlaps sky region then it's likely to be a tree
- - if an image depicts an old tree and a young sampling then it could be a metaphor of a life cycle

Description of Sets

- Belongingness (inclusion)
  - *-enumerate (characterize) elements belonging to A*
- Characteristic function

$$A : \mathbf{X} \rightarrow \{0,1\}$$



Sets :
Characteristic Functions

$$x \in A \iff A(x) = 1$$

$$x \notin A \iff A(x) = 0$$

Sets subscribe to the concept of **dichotomy**

Fuzzy Sets :
Membership Functions

- Admit a notion of partial membership of element to the concept
- the *higher* the membership value A(x), the more *typical* x is in A

Fuzzy Sets : A Definition

- Fuzzy set A is characterized by a membership function

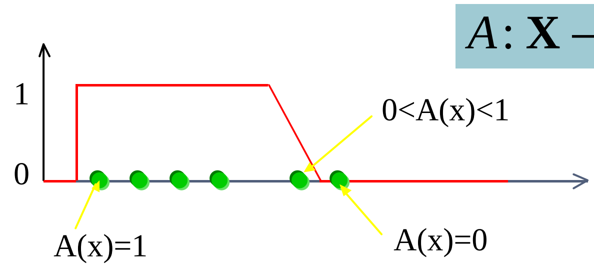
$$A : \mathbf{X} \rightarrow [0,1]$$

$A(x) = 1$	complete membership
$0 < A(x) < 1$	partial membership
$A(x) = 0$	complete exclusion

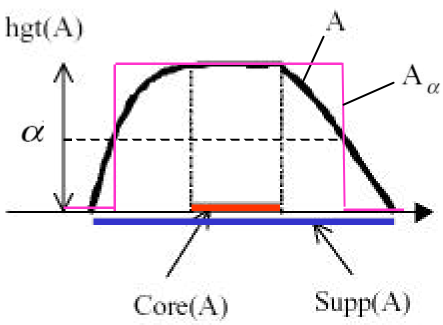
Description of Fuzzy Sets

- Membership function
  - *Characterize degree of belonging to A (degree of similarity, degree of preference)*

$$A : \mathbf{X} \rightarrow [0,1]$$



Geometry of Fuzzy Sets



$\text{Core}(A) = \{x \in \mathbf{X} \mid A(x) = 1\}$

$\text{Supp}(A) = \{x \in \mathbf{X} \mid A(x) > 0\}$

$A_\alpha = \{x \in \mathbf{X} \mid A(x) \geq \alpha\}$

The Representation Theorem

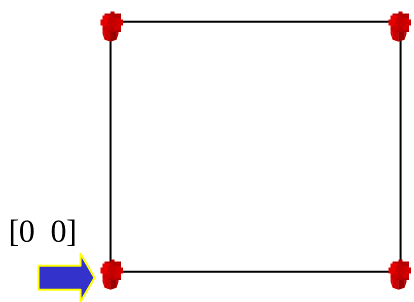
Fuzzy set A is *represented* by an infinite family of it's α-cuts

$$A = \cup_{a \in (0,1]} (a A_a)$$

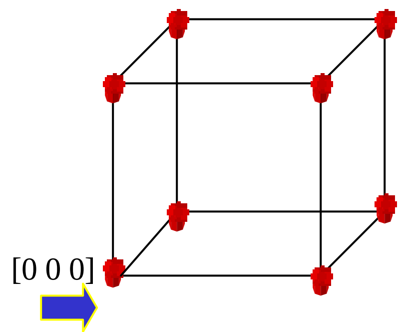
In terms of membership functions fuzzy set is *represented as*

$$A = \sup_{a \in [0,1]} (a A_a)$$

Geometry of Sets

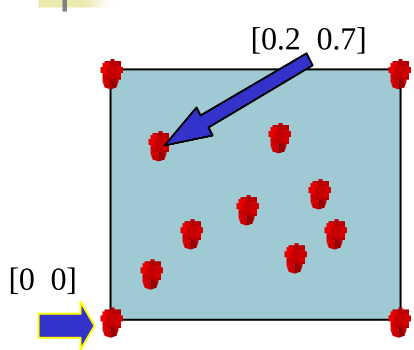


$\mathbf{X} = \{x_1, x_2\}$

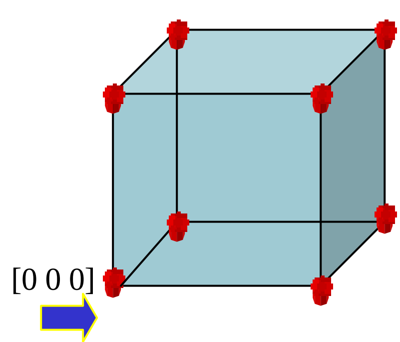


$\mathbf{X} = \{x_1, x_2, x_3\}$

Geometry of Fuzzy Sets



$\mathbf{X} = \{x_1, x_2\}$



$\mathbf{X} = \{x_1, x_2, x_3\}$

Operations on Fuzzy Sets

- Union

$$(A \cup B)(x) = \max(A(x), B(x))$$

- Intersection

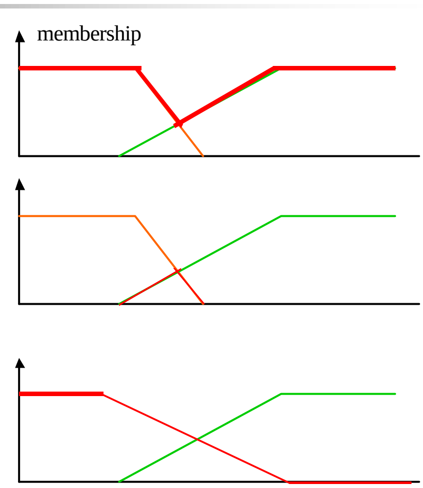
$$(A \cap B)(x) = \min(A(x), B(x))$$

- Complement

$$\overline{A}(x) = 1 - A(x)$$

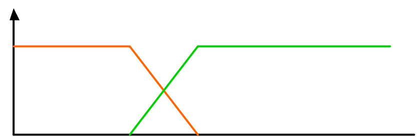
Operations on Fuzzy Sets

- Union
- Intersection
- Complement

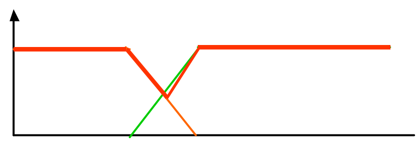


Overlap and underlap of Fuzzy Sets

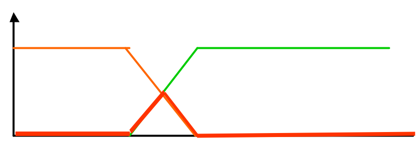
- A and its complement $\overline{A}$



- $A \cup \overline{A}$



- $A \cap \overline{A}$



Properties of Sets

Involution	$\overline{\overline{A}} = A$	
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cap A = A$	$A \cup A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Identity	$A \cup \varnothing = A$	$A \cap \mathbf{X} = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$	
Law of excluded middle	$A \cup \overline{A} = \mathbf{X}$	
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Properties of Fuzzy Sets

Involution	$\overline{\overline{A}} = A$	
Commutativity	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cap A = A$	$A \cup A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Identity	$A \cup \varnothing = A$	$A \cap \mathbf{X} = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$	**DO NOT hold !!!**
Law of excluded middle	$A \cup \overline{A} = \mathbf{X}$	
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Overlap and underlap of Fuzzy Sets

- Law of contradiction does *not* hold: overlap property

$$A \cap \overline{A} \supseteq \varnothing$$

- Law of excluded middle does *not* hold: underlap property

$$A \cup \overline{A} \subseteq \mathbf{X}$$

Dichotomy Problem

"… the law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are. …"

Russel, 1923

Generalisation of Operations on Fuzzy Sets

t-norms $t: [0,1] \times [0,1] \rightarrow [0,1]$

a t b = b t a	commutativity
a t (b t c) = (a t b) t c	associativity
if a<a' and b<b' then a t b a' t b'	monotonicity
at 0 = 0 a t 1 =a	boundary condition

s-norms $s: [0,1] \times [0,1] \rightarrow [0,1]$

a s b = b s a	commutativity
a s (b s c) = (a s b) s c	associativity
if a<a' and b<b' then a s b a' s b'	monotonicity
a s 0 = a a s 1 =1	boundary condition

Generalisation of Operations on Fuzzy Sets

t-norm	s-norm
min (x,y)	max(x,y)
xy	x+y-xy
$\max(0, (1+p)(x+y-1)-pxy), p \geq -1$	$\min(1, x+y+pxy),), p \geq -1$
$1-\min(1, \sqrt[p]{(1-x)^p+(1-y)^p}$ $p \geq 0$	$\min(1, \sqrt[p]{x^p+y^p}$ $p \geq 0$
$\frac{xy}{p+(1-p)(x+y-xy)}, \quad p \geq 0$	$\frac{x+y-xy-(1-p)xy}{1-(1-p)xy}, \quad p \geq 0$
$\frac{1}{\sqrt[p]{\frac{1}{x^p}+\frac{1}{y^p}-1}}, \quad p > 0$	$1-\frac{1}{\sqrt[p]{\frac{1}{(1-x)^p}+\frac{1}{(1-y)^p}-1}}, \quad p > 0$
$\frac{xy}{\max(xy, p)}, \quad p \in [0,1]$	$1-\frac{(1-x)(1-y)}{\max((1-x)(1-y), p)}, \quad p \in [0,1]$
$\begin{cases} x \text{ if } y = 1 \\ y \text{ if } x = 1 \\ 0, \text{ otherwise} \end{cases}$	$\begin{cases} x \text{ if } y = 0 \\ y \text{ if } x = 0 \\ 1, \text{ otherwise} \end{cases}$

Information granularity and Fuzzy Sets

Energy measure of fuzziness

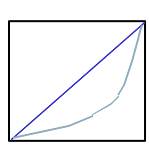
$$E(A) = \int_X e(A(x))dx$$

e: [0,1] $\rightarrow$ [0, 1] is an increasing function

$e(u) = u$

$e(u) = u^p$

$e(u) = \sin(\pi u/2)$



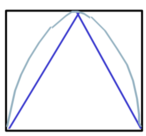
(weighted sum of elements in a fuzzy set)

Information granularity and Fuzzy Sets

Entropy measure of fuzziness

$$H(A) = \int_X h(A(x))dx$$

$h(u)$: $[0,1] \rightarrow [0, 1]$ continuous and $h(u)=h(1-u)$



$$h(u)=\begin{cases} 2u & \text{if } u \in [0, 1/2] \\ 2(1-u) & \text{if } u \in [1/2, 1] \end{cases}$$

$$h(u) = -u\log(u) - (1-u)\log(1-u)$$

(weighted sum of hesitation about elements in a fuzzy set)

Comparison of Fuzzy Sets

- distance measures
- possibility measure
- necessity measure
- compatibility measure

Comparison of Fuzzy Sets: Distance measure

$$d(A,B) = \sqrt[p]{\int_X |A(x) - B(x)|^p \, dx} \quad p \geq 1$$

$\text{Ham min g distance} : p = 1$

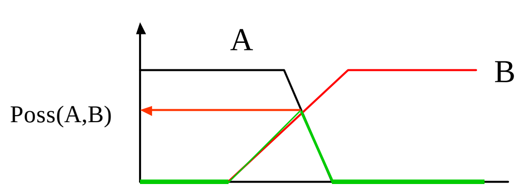
$\text{Euclidean distance} : p = 2$

....

$\text{Tchebysche v distance} : p = \infty$

Comparison of Fuzzy Sets: Possibility measure

$$Poss(A,B) = \sup_{x \in X}[\min(A(x), B(x))]$$



degree of overlap between A and B

Possibility measure: Main properties

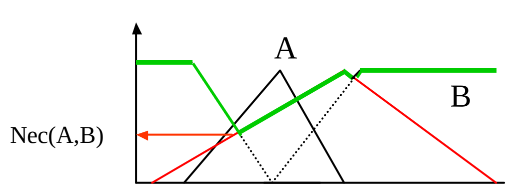
$$\mathrm{Poss}(A,B)=\sup_{x\in \mathbf{X}}[\min(A(x),B(x))]$$

$$\mathrm{Poss}(A\cup B,C)=\max[\mathrm{Poss}(A,C),\mathrm{Poss}(B,C)]$$

$$\mathrm{Poss}(\{x\},B)=B(x)$$

Comparison of Fuzzy Sets: Necessity measure

$$\mathrm{Nec}(A,B)=\inf_{x\in \mathbf{X}}[\max(1-A(x),B(x))]$$



degree of inclusion of A in B

Nesessity measure: Main properties

$$Nec(A,B) = \inf_{x \in X}[\max(1 - A(x), B(x))]$$

Nec(A, X)=1   Nec(X, A)=0

Nec(A, A)= 0.5

Nec({x}, A)= A(x)

Comparison of Fuzzy Sets: Compatibility measure

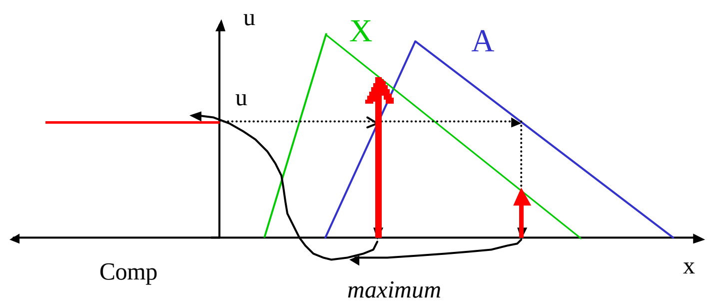
X, A: fuzzy sets defined in **X**

$$Comp(X, A)(u) = \sup_{x:u=A(x)} X(x); \quad u \in [0,1]$$

Comp(X,A)= Compatibility (X is A)

Comparison of Fuzzy Sets: Compatibility measure

$$\mathrm{Comp}(X, A)(u) = \sup_{x:u=A(x)} X(x); \quad u \in [0,1]$$



Comparison of Fuzzy Sets: Compatibility measure

$X, A: normal$ fuzzy sets

- if $\mathrm{X} \subset \mathrm{X'}$ then $\mathrm{Comp}(X, A)(u) \leq \mathrm{Comp}(X', A)(u)$
- if $\mathrm{X} = \mathrm{A}$ then $\mathrm{Comp}(X, X)(u) = u$
- if $\mathrm{X} = \{x_0\}$ then $\mathrm{Comp}(X, A) = \begin{cases} 1, \text{if } u = A(x_0) \\ 0, otherwise \end{cases}$

Comparison of Fuzzy Sets: Equality index

consider two fuzzy sets A and B in the same finite space $\mathbf{X} = \{x_1, x_2, ..., x_n\}$

$$A \equiv B = (A \subset B) \,\&\, (B \subset A)$$

(recall set theory)

- how to model inclusion $(\subset)$
- inclusion and implication $(\rightarrow)$

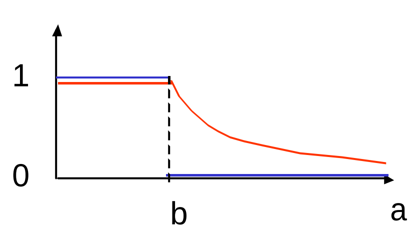
Inclusion and implication

$a, b \in [0,1]$

Examples of implications

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \le b \\ 0 & \text{if } a > b \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \le b \\ b/a & \text{if } a > b \end{cases}$$

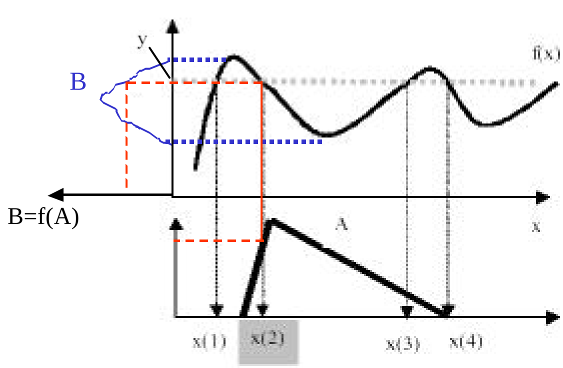


Comparison of Fuzzy Sets: Equality index

$$A \equiv B = \frac{1}{n}\sum_{i=1}^{n} \min[A(x_i) \rightarrow B(x_i)),(B(x_i) \rightarrow A(x_i))]$$

Transformations of Fuzzy Sets

Extension principle



The Extension Principle

B = f(A)

$B(y) = \sup_{x:\, y=f(x)} [A(x)]$

$B(y) = \sup A(x)$

subject to

$y = f(x)$

The Extension Principle: Example

X={a, b, c, d, e}

$f: A \rightarrow B$

$f(a)=\beta$, $f(b)=\alpha$, $f(c)=\chi$, $f(d)=\alpha$, $f(e)=\delta$

A(a)=1, A(b)=0.6, A(c)=0.2, A(d)=0.7, A(e)=0.9

$B(\beta)=A(a)=1$,

$B(\alpha)=\max(A(b), A(d))=0.7$

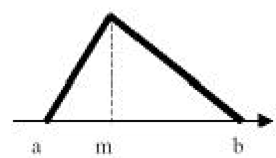
$B(\chi)=A(c)=0.2$,

$B(\delta)=A(e)=0.9$

Fuzzy Arythmetic

Fuzzy number

A(a, m, b)



Addition of fuzzy numbers

$$C = A \oplus B$$

$$C(z) = \sup_{x,y\in\mathbf{R}: z=x+y}[\min(A(x),\ B(y))] = \sup_x[\min(A(x),\ B(z-x))]$$

Fuzzy Arythmetic

Fuzzy multiplication, division etc. do not preserve the linearity of the membership function

In general the use of a-cuts and The Representation Theorem results in a *brute force* method of carrying out computing with fuzzy numbers